The triviality of tangent bundle

We present the proof of John Milnor on the following.

Theorem 0.1. *The tangent bundle $T\mathbb{S}^2$ is non-trivial.*

It follows from the following famous Theorem in differential topology.

Theorem 0.2 (simplified version)**.** *Suppose v is a smooth vector field on $\mathbb{S}^2$, then X vanishes somewhere.*

Proof. By the natural embedding, we can identify $T\mathbb{S}^2$ as $\{v \in \mathbb{R}^3 : \langle v, x\rangle = 0, \forall x \in \mathbb{S}^2\}$ where x is understood as the position vector.

Suppose there is smooth vector field $v : \mathbb{S}^2 \to T\mathbb{S}^2$ such that $|v| = 1$ by rescaling. Consider the map $F_t : \mathbb{S}^2 \to \mathbb{S}^2_{\sqrt{1+t^2}}$ given by

$$F_t(x) = x + tv(x).$$

We note that since v is smooth, if $F_t(x) = F_t(y)$, then

$$|x - y| = t|v(x) - v(y)| \le Ct|x - y| \tag{1}$$

which implies F_t is injective if t is sufficiently small. Extend $v(x)$ on Annulus $A(r, R)$ by $\tilde{v}(x) = |x| \cdot v\left(\frac{x}{|x|}\right)$ for some fixed $r < 1 < R$. And we extend the map F_t to $A(r, R)$ by $F_t(x) = x + t\tilde{v}(x)$ for $x \in A(r, R)$.

We claim that $F_t(\mathbb{S}^2) = \mathbb{S}^2_{\sqrt{1+t^2}}$. If so, then $F_t(A(r, R)) = \sqrt{1 + t^2} \cdot A(r, R)$ by the scaling properties of F. Assuming this is true, then

$$\begin{aligned} \mathrm{Vol}_{euc}\left(\sqrt{1+t^2} \cdot A(r,R)\right) &= \mathrm{Vol}_{euc}\left(F_t(A(r,R))\right) \\ &= \int_{A(r,R)} |dF_t| d\mu \end{aligned} \tag{2}$$

where the left hand side is of $(1 + t^2)^{3/2} \cdot C$ while

$$(F_t)^i_j = \delta^i_j + t \cdot \tilde{v}^i_j$$

and hence the integral is in form of polynomial of t which is impossible. Noted that the C^1 properties of $\tilde{v}$ is nothing but from v (by scaling).

Mistake made in class: the change of coordinate formula is true but not as nice as the above stated form. This is because in local coordinate of sphere, F_t is a mess. I over-thought this part.

It suffices to prove the claim. The inclusion is trivial, it remains to prove the surjective. Since F_t is smooth, $F_t(\mathbb{S}^2)$ is compact and hence closed. We claim that F_t is a open map on $A(r, R)$. Let U be a open set and $y = F_t(x)$ for some $x \in U$. Since $dF_t \ne 0$ on $A(r, R)$, Inverse function Theorem implies that F_t has a smooth inverse around x which in particular implies $F_t(U)$ is open. And hence, $F_t(\mathbb{S}^2)$ is relatively open in $\mathbb{S}^2_{\sqrt{1+t^2}}$. This proves the claim by connectedness.

It is not difficult to see from the proof that 1. the dimension is not necessarily 2, 2. the regularity of v is not necessarily smooth.

□